An example concerning Ohtsuki's invariant and the full $SO(3)$ quautum invariant *

Bang-He Li & Tian-Jun Li

Abstract

Two lens spaces are given to show that Ohtsuki's τ for rational homology spheres does not determine Kirby-Melvin's $\{\tau_r', r \text{ odd} \geq 3\}$

By using partial Kirby-Melvin's quautum $SO(3)$ invariants $\{\tau_r'(M), r \text{ odd prime} > |H_1(M,Z)|\}$ [KM], Ohtsuki [O] defined a topological invariant

$$\tau(M) = \sum_{n=0}^{\infty} \lambda_n(M)(t-1)^n \in Q[[t-1]]$$

for rational homology 3-sphere M. R.Lawrence [La] conjectured that

$$\lambda_n(M) \in \begin{cases} Z, & \text{if } |H_1(M,Z)| = 1 \\ Z[\frac{1}{2}, \frac{1}{|H_1(M,Z)|}], & \text{if } |H_1(M,Z)| > 1 \end{cases}$$

and if r is an odd prime which does not divide $|H_1(M,Z)|$, then $\{|H_1(M,Z)|\}_r \tau_r'(M)$ is the $r-$adic limit of the series

$$\sum_{n=0}^{\infty} \lambda_n^{\vee}(M) h^n$$

where $\{\cdot\}_r$ stands for the Jacobi symbol, and $h = e^{\frac{2\pi i}{r}} - 1$.

Rozansky [R] has proved that this conjecture is true. So $\tau(M)$ and $\{\tau_r', r \text{ odd prime}$ not dividing $|H_1(M,Z)|\}$ determine each other.

A natural question arises: Does τ determine all $\{\tau_r', r \text{ odd} \geq 3\}$?

It was proved in [Li] that $\tau_r'(M) = \tau_r'(M')$ iff $\xi_r(M,A) = \xi_r(M',A)$ for r odd ≥ 3, where A is any $r-$th primitive root of unit. So the question is equivalent to: Does $\tau(M)$ determine all $\xi_r(M,e_r)$? Where, e_a stands for $e^{\frac{2\pi i}{a}}$.

*The first author is supported partially by the National Science Foundation of P. R. China, the second author is supported partially by NSF grant DMS 9304580

For lens space $L(p,q)$, all $\xi_r(L(p,q),e_r)$ has been obtained in [LL1] (explicit formulas for $\tau_r^{'}(L(p,q))$ were given in [LL2]), that is: let $r\geq 3$ be odd and $c=(p,r)$ the common factor, then

$$(1)\quad \xi_r(L(p,q),e_r)=\begin{cases}\{p\}_r e_r^{-12s(q,p)}e_p^{r'(q+q^*)}\dfrac{e_r^{2p'}-e_r^{-2p'}}{e_r^2-e_r^{-2}}\ , & \text{if } c=1\\[2ex] (-1)^{\frac{r-1}{2}\frac{c-1}{2}}\{p/c\}_{r/c}\{q\}_c e_r^{-12s(q,p)} & \\ e_{pc}^{(r/c)'(q+q^*-\eta p^*p)}e_{rc}^{-2\eta(p/c)'}\dfrac{\epsilon(c)\sqrt{c}\eta}{e_r^{-2}-e_r^2}\ , & \text{if } c>1\ ,\ c\mid q^*+\eta\\ 0, & \text{if } c>1 \text{ and } c\not|q^*\pm 1\end{cases}$$

where $\eta=1$ or -1, $p^*p+q^*q=1$ with $0<q^*<p$, $(p/c)^{'}p/c+(r/c)'r/c=1$, $s(p,q)$ is the Dedekind sum, and

$$\epsilon(c)=\begin{cases}1, & \text{if } c\equiv 1 \pmod 4\\ i, & \text{if } c\equiv -1 \pmod 4\end{cases}$$

Since

$$\tau(L(p,q))=t^{-3s(q,p)}\frac{t^{\frac{1}{2p}}-t^{-\frac{1}{2p}}}{t^{\frac{1}{2}}-t^{-\frac{1}{2}}}$$

([O] and [LL2]), $\tau(L(p_1,q_1))=\tau(L(p_2,q_2))$ iff $p_1=p_2$ and $s(q_1,p_1)=s(q_2,p_2)$. The following Theorem answers the question above.

Theorem. $s(6,25)=s(11,25)$, while $\xi_r(L(25,11))=\xi_r(L(25,6))$ if and only if $(r,25)\neq 5$.

Proof. We calculate $s(q,p)$ by the formula in [H]:

$$12s(q,p)=\sum_{i=1}^{n}m_i+\frac{q+q^*}{p}-3n$$

if $\frac{p}{q}=m_n-\cfrac{1}{m_{n-1}-\cdots-\cfrac{1}{m_2-\cfrac{1}{m_1}}}$ with $m_i\geq 2$.

Now

$$\frac{25}{6}=5-\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2}}}}},\quad \frac{25}{11}=3-\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{3-\cfrac{1}{2}}}}$$

$11^*=16, 6^*=21$, so

$$12s(6,25)=12s(11,25)=-3+\frac{27}{25}$$

$c=(r,25)$ can be only $1,5$ or 25. If $c=25$, then $c \not| 6^* \pm 1$ and $c \not| 11^* \pm 1$, thus by (1) $\xi_r(L(25,11),e_r)=\xi_r(L(25,6),e_r)=0$. If $c=1$, it is easy to see from (1) that the two ξ_r are equal.

Assume $c=5$, then $c|6^*-1$ and 11^*-1. Now since $q^*q+p^*p=1$, we have

$$q+q^*+(25)^*\times 25=\begin{cases} 27+125, & \text{if } q=6 \\ 27+175, & \text{if } q=11 \end{cases}$$

Therefore, by (1)

$$\frac{\xi_r(L(25,11),e_r)}{\xi_r(L(25,6),e_r)}=e_{125}^{(r/5)'\times 50}$$

Since $(p/c)'p/c+(r/c)'r/c=1$, we see that $(r/5)'$ is prime to $(p/5)=5$. This shows that $e_{125}^{(r/5)'\times 50}\neq 1$, and the theorem is proved.

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Author's address

Bang-He Li	Tian-Jun Li
Institute of Systems Science,	School of math.
Academia Sinica	IAS
Beijnig 100080	Princeton NJ 08540
P. R. China	U.S.A.
Libh@iss06.iss.ac.cn	Tjli@IAS.edu